

# Theory of Josephson effect in chiral $p$ -wave superconductor / diffusive normal metal / chiral $p$ -wave superconductor junctions

Y. Sawa<sup>1</sup>, T. Yokoyama<sup>1</sup>, Y. Tanaka<sup>1</sup>, A. A. Golubov<sup>2</sup>

<sup>1</sup>*Department of Applied Physics, Nagoya University, Nagoya, 464-8603, Japan*

*and CREST, Japan Science and Technology Corporation (JST) Nagoya, 464-8603, Japan*

<sup>2</sup>*Faculty of Science and Technology, University of Twente, 7500 AE, Enschede, The Netherlands*

(Dated: February 6, 2008)

We study the Josephson effect between chiral  $p$ -wave superconductor / diffusive normal metal (DN) / chiral  $p$ -wave superconductor (CP/DN/CP) junctions using quasiclassical Green's function formalism with proper boundary conditions. The  $p_x + ip_y$ -wave symmetry of superconducting order parameter is chosen which is believed to be a pairing state in  $\text{Sr}_2\text{RuO}_4$ . It is shown that the Cooper pairs induced in DN have an odd-frequency spin-triplet  $s$ -wave symmetry, where pair amplitude is an odd function of Matsubara frequency. Despite the peculiar symmetry properties of the Cooper pairs, the behavior of the Josephson current is rather conventional. We have found that the current phase relation is almost sinusoidal and the Josephson current is proportional to  $\exp(-L/\xi)$ , where  $\xi$  is the coherence length of the Cooper pair in DN and  $L$  is the length of DN. The Josephson current between CP / diffusive ferromagnet metal (DF) / CP junctions is also calculated. It is shown that the  $0-\pi$  transition can be realized by varying temperature or junction length  $L$  similar to the case of conventional  $s$ -wave junctions. These results may serve as a guide to study superconducting state of  $\text{Sr}_2\text{RuO}_4$ .

## I. INTRODUCTION

Exploration of unconventional superconducting pairing is one of central issues in the physics of superconductivity. Possible realization of spin-triplet superconductivity in  $\text{Sr}_2\text{RuO}_4$  is widely discussed at present<sup>1</sup>. A number of experimental results is consistent with spin-triplet chiral  $p$ -wave symmetry state in this material<sup>2,3,4</sup>. It is well known that the midgap Andreev resonant state (MARS)<sup>5,6,7,8</sup> is induced near interfaces in unconventional superconducting junctions where pair potential changes its sign across the Fermi surface. The MARS manifests itself as a zero bias conductance peak (ZBCP) in quasiparticle tunneling experiments. A number of tunneling data in  $\text{Sr}_2\text{RuO}_4$  junctions show ZBCP<sup>9</sup> in accordance with theoretical predictions<sup>10</sup>. The Josephson effect in  $\text{Sr}_2\text{RuO}_4$  was also studied both theoretically<sup>11</sup> and experimentally<sup>12</sup>. Recent SQUID experiment by Nelson<sup>13</sup> is consistent with the realization of chiral  $p$ -wave superconducting state<sup>14</sup>.

At the same time, there are important recent achievements in theoretical study of the proximity effect in junctions between unconventional superconductors. It was predicted that in diffusive normal metal (DN) / triplet superconductor (TS) junctions MARS formed at the DN/TS interface, can penetrate into DN<sup>15</sup>. This proximity effect is very unusual since it generates the zero energy peak (ZEP) in the local density of states (LDOS) in contrast to the conventional proximity effect where the resulting LDOS has a minigap<sup>16</sup>. It was also shown theoretically, that the ZEP appears in the chiral  $p$ -wave state. Thus to explore the ZEP in DN region of DN/ $\text{Sr}_2\text{RuO}_4$  heterostructures is an intriguing topic<sup>15</sup>. Very recently, it is predicted that the induced Cooper pairs in DN are in an unconventional odd-frequency symmetry state, in contrast to the usual even-frequency pairing<sup>17</sup>. Since this

proximity effect specific to TS is completely new phenomenon, it is very interesting to study the Josephson effect in TS/DN/TS junctions.

Recently, it is shown that the Josephson current is enhanced strongly at low temperatures in TS/DN/TS junctions<sup>18</sup> and is proportional to  $\sin(\Psi/2)$ , where  $\Psi$  is a superconducting phase difference between left and right superconductors<sup>19</sup>. These results are quite different from those for  $d$ -wave superconductor / DN /  $d$ -wave superconductor junctions<sup>20</sup>. However, in most of previous theories of TS/DN/TS junctions, only the  $p$ -wave state in the presence of the time reversal symmetry was considered. The existing knowledge of the Josephson effect in TS/DN/TS junctions for chiral  $p$ -wave symmetry is very limited<sup>21</sup>. It is important to study the Josephson effect in the chiral  $p$ -wave junctions in much more detail because this symmetry is the most promising superconducting state in  $\text{Sr}_2\text{RuO}_4$ .

To study this problem, the quasi-classical Green's function theory is the useful method. In diffusive regime, the quasi-classical Green's function obeys the Usadel equations<sup>22</sup>. The circuit theory<sup>23</sup> enables one to treat the case of arbitrary interface transparency in  $s$ -wave superconductor (S) junctions. This theory was recently generalized for unconventional superconductor (US) junctions<sup>24,25,26,27</sup>. In these approach, the effect of MARS is naturally included. The theory was extended to the cases of US/DN/US and US/diffusive ferromagnet (DF)/US junctions where time reversal symmetry is present in US<sup>28,29</sup>. However, these theories can not be applied to calculating the Josephson current in chiral  $p$ -wave superconductor / DN / chiral  $p$ -wave superconductor (CP/DN/CP) junctions with  $p_x + ip_y$ -wave symmetry of the pair wave function in chiral  $p$ -wave superconductor. The purpose of this paper is to generalize the above approach and apply it to the interface between DN (DF)

/ superconductor with broken time reversal symmetry.

In the present paper, we derive the boundary conditions of quasiclassical Green's function available for DN (DF) / CP interface in the presence of the Josephson effect and calculate the Josephson current in CP / DN (DF) / CP junctions by solving the Usadel equations with these boundary conditions. It is shown that the induced pair in DN is purely in the odd-frequency pairing state. The magnitude of the calculated Josephson current is larger than that in the  $s$ -wave superconductor / DN /  $s$ -wave superconductor (S/DN/S) junctions. However, it is smaller than that in  $p_x$ -wave superconductor / DN /  $p_x$ -wave superconductor (P/DN/P) junctions. The obtained temperature dependence of the Josephson current is similar to that in the conventional  $s$ -wave junctions. The current phase relation is almost sinusoidal and the Josephson current is proportional to  $\exp(-L/\xi)$ , where  $\xi$  is the coherence length of the Cooper pair in DN and  $L$  is the width of DN. We have also calculated the Josephson current in CP/DF/CP junctions. Similar to the case of the S/DF/S junctions, the  $0 - \pi$  transition occurs as a function of the length of DF. As follows from these results, it is difficult to extract the unusual properties of proximity effect specific to spin-triplet  $p$ -wave superconductor junctions if we look at d.c. Josephson effect only. These results may serve as a guide to explore novel properties in superconducting  $\text{Sr}_2\text{RuO}_4$ .

## II. FORMULATION

In the following sections, the units with  $\hbar = k_B = 1$  are used. The model of CP / DF (DN) / CP junction is illustrated in Fig.1. Here  $R'_b$  is a resistance of insulating barrier located at  $x = 0$ ,  $R_d$  is a resistance of the DN,  $R_b$  is a resistance of insulating barrier located at  $x = L$ , and the length of DN  $L$  is much larger than the mean free path. The infinitely narrow insulating barriers are modeled as  $U(x) = H'\delta(x) + H\delta(x - L)$ . Then the barrier transparency  $T_m^{(\prime)}$  is expressed by  $T_m^{(\prime)} = 4 \cos^2 \phi / (4 \cos^2 \phi + Z^{(\prime)2})$  with  $Z^{(\prime)} = 2H^{(\prime)}/v_F$ . Here  $\phi$  is injection angle measured from the direction perpendicular to the interface between DF (DN) and chiral superconductor, and  $v_F$  is Fermi velocity.

First, we concentrate on the Nambu-Keldysh (NK) Green's function in DF (DN) within the quasiclassical approximation. We define NK Green's function as  $\check{G}_N(x)$ . We denote the retarded part of  $\check{G}_N(x)$  as  $\hat{R}_N(x)$ , which is given by

$$\hat{R}_N = \sin \theta \cos \psi \hat{\tau}_1 + \sin \theta \sin \psi \hat{\tau}_2 + \cos \theta \hat{\tau}_3, \quad (1)$$

where  $\hat{\tau}_j$  ( $j=1,2,3$ ) are the Pauli matrices in electron-hole space.

The functions  $\theta$  and  $\psi$  for majority (minority) spins obey the Usadel equation:

$$D \left[ \frac{\partial^2}{\partial x^2} \theta - \left( \frac{\partial \psi}{\partial x} \right)^2 \cos \theta \sin \theta \right] + 2i(\varepsilon + (-)h) \sin \theta = 0,$$

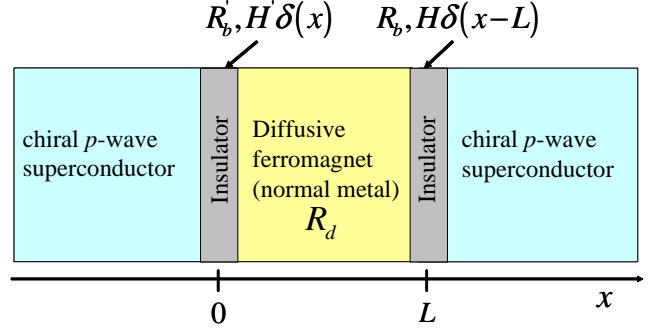


FIG. 1: (Color online) Schematic illustration of the model.

$$\frac{\partial}{\partial x} \left[ \sin^2 \theta \left( \frac{\partial \psi}{\partial x} \right) \right] = 0, \quad (2)$$

with the diffusion constant  $D$  and the exchange field  $h$ . If we choose  $h = 0$ , DF is reduced to be DN. The boundary condition for  $\check{G}_N(x)$  at DF (DN) / CP interface has the form

$$\frac{L}{R_d} \left[ \check{G}_N(x) \frac{\partial \check{G}_N(x)}{\partial x} \right] \Big|_{x=L_-} = -\frac{\langle \check{I}_m \rangle}{R_b},$$

$$\check{I}(\phi) = \check{I}_m = 2 [\check{G}_1, \check{B}_m],$$

$$\begin{aligned} \check{B}_m = & (-T_{1m}[\check{G}_1, \check{H}^{-1}] + \check{H}^{-1} \check{H}_+ - T_{1m}^2 \check{G}_1 \check{H}^{-1} \check{H}_+ \check{G}_1)^{-1} \\ & \times (T_{1m}(1 - \check{H}^{-1}) + T_{1m}^2 \check{G}_1 \check{H}^{-1} \check{H}_+), \end{aligned} \quad (3)$$

with  $\check{G}_1 = \check{G}_N(x = L_-)$ ,  $\check{H}_\pm = (\check{G}_{2+} \pm \check{G}_{2-})/2$ , and  $T_{1m} = T_m/(2 - T_m + 2\sqrt{1 - T_m})$ . Here  $\check{G}_{2\pm}$  is the asymptotic Green's function in the superconductor defined as in the previous paper<sup>24</sup>.

Here, the average over the various angles of injected particle at the interface is defined as

$$\langle \check{I}_m \rangle = \frac{\int_{-\pi/2}^{\pi/2} d\phi \cos \phi \check{I}_m}{\int_{-\pi/2}^{\pi/2} d\phi \cos \phi T_m}. \quad (4)$$

Then the resistance of the interface  $R_b^{(\prime)}$  is given by

$$R_b^{(\prime)} = \frac{2R'_0}{\int_{-\pi/2}^{\pi/2} d\phi \cos \phi T_m^{(\prime)}}, \quad (5)$$

where  $R_0^{(\prime)}$  is the Sharvin resistance, which in the three dimensional case is expressed by  $R_0^{(\prime)-1} = e^2 k_F^2 S'_c / 4\pi^2$ . Here,  $k_F$  is the Fermi wave number and  $S'_c$  is the constriction area.

Retarded component of  $\check{G}_{2\pm}$ , i.e.,  $\hat{R}_{2\pm}$ , is expressed by

$$\begin{aligned} \hat{R}_{2\pm} = & (f_{1\pm} \cos \Psi + f_{2\pm} \sin \Psi) \hat{\tau}_1 \\ & + (f_{1\pm} \sin \Psi - f_{2\pm} \cos \Psi) \hat{\tau}_2 + g_{\pm} \hat{\tau}_3, \end{aligned} \quad (6)$$

with  $f_{1\pm} = \text{Re}(f_{\pm})$ ,  $f_{2\pm} = \text{Im}(f_{\pm})$ ,  $g_{\pm} = \epsilon / \sqrt{\epsilon^2 - |\Delta_{\pm}|^2}$ ,  $f_{\pm} = \Delta_{\pm} / \sqrt{|\Delta_{\pm}|^2 - \epsilon^2}$ , and the macroscopic phase of the superconductor  $\Psi$ . Here,  $\Delta_+ = \Delta(\phi)$  and  $\Delta_- = \Delta(\pi - \phi)$  are the pair potentials corresponding to the injection angles  $\phi$  and  $\pi - \phi$  respectively. Note that  $|\Delta_+| = |\Delta_-|$  is satisfied in the present case, then we can put  $g_+ = g_- \equiv g$ .

Next we consider the boundary condition for the retarded part of the NK Green's functions at the DF (DN) / CP interface. The left side of the boundary condition of Eq.(3) is expressed by

$$\begin{aligned} & \frac{L}{R_d} \hat{R}_N(x) \frac{\partial}{\partial x} \hat{R}_N(x)|_{x=L_-} \\ &= \frac{Li}{R_d} \left[ \left( -\frac{\partial \theta}{\partial x} \sin \psi - \frac{\partial \psi}{\partial x} \sin \theta \cos \theta \cos \psi \right) \hat{\tau}_1 \right. \\ & \left. + \left( \frac{\partial \theta}{\partial x} \cos \psi - \frac{\partial \psi}{\partial x} \sin \theta \cos \theta \sin \psi \right) \hat{\tau}_2 + \frac{\partial \psi}{\partial x} \sin^2 \theta \hat{\tau}_3 \right]. \end{aligned} \quad (7)$$

In the right side of Eq.(3),  $\hat{I}_R$  is expressed by

$$\begin{aligned} \hat{I}_R &= 4iT_{1m}(\mathbf{d}_R \cdot \mathbf{d}_R)^{-1} \\ &\times \left( -\frac{1}{2}(1 + T_{1m}^2)(\mathbf{s}_{2+} - \mathbf{s}_{2-})^2 [\mathbf{s}_1 \times (\mathbf{s}_{2+} + \mathbf{s}_{2-})] \cdot \hat{\tau} \right. \\ &+ 2T_{1m}\mathbf{s}_1 \cdot (\mathbf{s}_{2+} \times \mathbf{s}_{2-}) [\mathbf{s}_1 \times (\mathbf{s}_{2+} \times \mathbf{s}_{2-})] \cdot \hat{\tau} \\ &+ 2T_{1m}\mathbf{s}_1 \cdot (\mathbf{s}_{2+} - \mathbf{s}_{2-}) [\mathbf{s}_1 \times (\mathbf{s}_{2+} - \mathbf{s}_{2-})] \cdot \hat{\tau} \\ &- i(1 + T_{1m}^2)(1 - \mathbf{s}_{2+} \cdot \mathbf{s}_{2-}) [\mathbf{s}_1 \times (\mathbf{s}_{2+} \times \mathbf{s}_{2-})] \cdot \hat{\tau} \\ &+ 2iT_{1m}(1 - \mathbf{s}_{2+} \cdot \mathbf{s}_{2-}) [\mathbf{s}_1 \cdot (\mathbf{s}_{2+} - \mathbf{s}_{2-})\mathbf{s}_1 \\ &\quad \left. - (\mathbf{s}_{2+} - \mathbf{s}_{2-})] \cdot \hat{\tau} \right), \end{aligned} \quad (8)$$

$$\mathbf{d}_R = (1 + T_{1m}^2)(\mathbf{s}_{2+} \times \mathbf{s}_{2-}) - 2T_{1m}\mathbf{s}_1 \times (\mathbf{s}_{2+} - \mathbf{s}_{2-}) - 2T_{1m}^2\mathbf{s}_1 \cdot (\mathbf{s}_{2+} \times \mathbf{s}_{2-})\mathbf{s}_1, \quad (9)$$

with  $\hat{R}_1 = \mathbf{s}_1 \cdot \hat{\tau}$ , and  $\hat{R}_{2\pm} = \mathbf{s}_{2\pm} \cdot \hat{\tau}$ . Here,  $\hat{I}_R$  is the retarded part of  $\hat{I}_m$ . The spectral vector  $\mathbf{s}_1$ , and  $\mathbf{s}_{2\pm}$  are expressed by

$$\mathbf{s}_1 = \begin{pmatrix} \sin \theta \cos \psi \\ \sin \theta \sin \psi \\ \cos \theta \end{pmatrix},$$

$$\mathbf{s}_{2\pm} = \begin{pmatrix} f_{1\pm} \cos \Psi + f_{2\pm} \sin \Psi \\ f_{1\pm} \sin \Psi - f_{2\pm} \cos \Psi \\ g \end{pmatrix}. \quad (10)$$

Here,  $\Psi$  is the macroscopic phase of right superconductor. After some algebra, the matrix current is reduced to be

$$\begin{aligned} \hat{I}_R &= 2iT_m[(2 - T_m) + T_m(g_s \cos \theta + f_s \sin \theta \sin(\psi - \Psi))]^{-1} \\ &\times ([-g_s \sin \theta \sin \psi - f_s \cos \theta \cos \Psi] \hat{\tau}_1 \\ &+ [g_s \sin \theta \cos \psi - f_s \cos \theta \sin \Psi] \hat{\tau}_2 \\ &+ f_s \sin \theta \cos(\psi - \Psi) \hat{\tau}_3) \end{aligned} \quad (11)$$

Then the resulting  $\hat{I}_R$  can be expressed as

$$\hat{I}_R = \begin{pmatrix} -I_1 \sin \theta \sin \psi - I_2 \cos \theta \cos \Psi \\ I_1 \sin \theta \cos \psi - I_2 \cos \theta \sin \Psi \\ I_2 \sin \theta \cos(\psi - \Psi) \end{pmatrix}, \quad (12)$$

$$I_1 = \left\langle \frac{2T_m g_s}{\Lambda_m} \right\rangle, I_2 = \left\langle \frac{2T_m f_s}{\Lambda_m} \right\rangle,$$

$$\Lambda_m = 2 - T_m + T_m [g_s \cos \theta - f_s \sin \theta \sin(\psi - \Psi)],$$

$$\begin{aligned} g_s &= \frac{2g + i(f_{1+}f_{2-} - f_{2+}f_{1-})}{1 + g^2 + f_{1+}f_{1-} + f_{2+}f_{2-}}, \\ f_s &= \frac{ig(f_{1+} - f_{1-}) + f_{2+} + f_{2-}}{1 + g^2 + f_{1+}f_{1-} + f_{2+}f_{2-}}. \end{aligned} \quad (13)$$

Finally the boundary condition is obtained as

$$\begin{aligned} \frac{LR_b}{R_d} \frac{\partial \theta}{\partial x} &= -I_1 \sin \theta - I_2 \cos \theta \sin(\psi - \Psi), \\ \frac{LR_b}{R_d} \frac{\partial \psi}{\partial x} \sin \theta &= -I_2 \cos(\psi - \Psi). \end{aligned} \quad (14)$$

For the calculation of thermodynamically quantities, it is convenient to use Matsubara representation by changing  $\epsilon \rightarrow i\omega$ .

We parameterize the quasiclassical Green's functions  $G_\omega$  and  $F_\omega$  with a function  $\Phi_\omega$ ,<sup>30,31</sup>

$$G_\omega = \frac{\omega}{\sqrt{\omega^2 + \Phi_\omega \Phi_{-\omega}^*}}, \quad F_\omega = \frac{\Phi_\omega}{\sqrt{\omega^2 + \Phi_\omega \Phi_{-\omega}^*}}, \quad (15)$$

where  $\omega$  is Matsubara frequency. The following relations are satisfied:

$$\begin{aligned} \frac{G_\omega}{2\omega} (\Phi_\omega + \Phi_{-\omega}^*) &= \sin \theta \cos \psi, \\ \frac{iG_\omega}{2\omega} (\Phi_\omega - \Phi_{-\omega}^*) &= \sin \theta \sin \psi. \end{aligned} \quad (16)$$

Then the Usadel equation for majority (minority) spin has the form<sup>31</sup>

$$\xi^2 \frac{\pi T_C}{G_\omega} \frac{\partial}{\partial x} \left( G_\omega^2 \frac{\partial}{\partial x} \Phi_\omega \right) - (\omega - (+)ih) \Phi_\omega = 0, \quad (17)$$

with the coherence length  $\xi = \sqrt{D/2\pi T_C}$ , the diffusion constant  $D$ , the exchange field  $h$ , and the transition temperature  $T_C$ .

The boundary condition at  $x = L$  is expressed by

$$\frac{G_\omega}{\omega} \frac{\partial}{\partial x} \Phi_\omega = \frac{R_d}{R_b L} \left( -\frac{\Phi_\omega}{\omega} I_1 + i e^{-i\Psi} I_2 \right),$$

$$I_1 = \left\langle \frac{2T_m g_s}{\Lambda_m} \right\rangle, I_2 = \left\langle \frac{2T_m f_s}{\Lambda_m} \right\rangle,$$

$$\Lambda_m = 2 - T_m + T_m [g_s G_\omega + f_s (B \sin \Psi - C \cos \Psi)],$$

$$B = \frac{G_\omega}{2\omega} (\Phi_\omega + \Phi_{-\omega}^*), C = \frac{i G_\omega}{2\omega} (\Phi_\omega - \Phi_{-\omega}^*). \quad (18)$$

The boundary condition at  $x = 0$  is expressed by

$$\frac{G_\omega}{\omega} \frac{\partial}{\partial x} \Phi_\omega = -\frac{R_d}{R_b' L} \left( -\frac{\Phi_\omega}{\omega} I_1' + i I_2' \right). \quad (19)$$

Here  $I_1'$  and  $I_2'$  are obtained by adding subscript ', changing  $\phi$  to  $\pi - \phi$ , and putting  $\Psi = 0$  for  $I_1$  and  $I_2$  at  $x = L$ . Then the macroscopic phase differences between left and right superconductor becomes  $\Psi$ .

To discuss the features of the proximity effect, in the following we will study the frequency dependence of the induced pair amplitude in DN by choosing  $h = 0$ . Before proceeding with formal discussion, let us present qualitative arguments. Two constrains should be satisfied in the considered system: (1) only the  $s$ -wave even-parity state is possible in the DN due to isotropization by impurity scattering, (2) the spin structure of induced Cooper pairs in the DN is the same as in an attached superconductor. Then the Pauli principle provides the unique relations between the pairing symmetry in a superconductor and the resulting symmetry of the induced pairing state in the DN<sup>17</sup>. Since there is no spin flip at the interface, it is natural to expect that the odd-frequency pairing state is generated in DN. It is possible to show that

$$f_\pm(-\omega) = f_\pm(\omega), \quad g_\pm(-\omega) = -g_\pm(\omega) \quad (20)$$

Using these equations,

$$g_s(-\omega, -\phi) = -g_s(\omega, \phi), \quad f_s(-\omega, -\phi) = -f_s(\omega, \phi). \quad (21)$$

are satisfied. For  $h = 0$ , the Usadel equation has the form<sup>31</sup>

$$\xi^2 \frac{\pi T_C}{G_\omega} \frac{\partial}{\partial x} \left( G_\omega^2 \frac{\partial}{\partial x} \Phi_\omega \right) - \omega \Phi_\omega = 0, \quad (22)$$

and the boundary condition at  $x = L$  is

$$\frac{G_\omega}{\omega} \frac{\partial}{\partial x} \Phi_\omega = \frac{R_d}{R_b L} \left( -\frac{\Phi_\omega}{\omega} I_1(\omega, \phi) + i e^{-i\Psi} I_2(\omega, \phi) \right),$$

$$I_1 = \left\langle \frac{2T_m g_s}{\Lambda_m} \right\rangle, I_2 = \left\langle \frac{2T_m f_s}{\Lambda_m} \right\rangle,$$

$$\Lambda_m = 2 - T_m + T_m [g_s G_\omega + f_s (B \sin \Psi - C \cos \Psi)],$$

$$B = \frac{G_\omega}{2\omega} (\Phi_\omega + \Phi_{-\omega}^*), C = \frac{i G_\omega}{2\omega} (\Phi_\omega - \Phi_{-\omega}^*). \quad (23)$$

By changing  $\omega$  and  $\phi$  into  $-\omega$  and  $-\phi$  in eqs. (23) and (24), following equations are obtained

$$\xi^2 \frac{\pi T_C}{G_{-\omega}} \frac{\partial}{\partial x} \left( G_{-\omega}^2 \frac{\partial}{\partial x} \Phi_{-\omega} \right) + \omega \Phi_{-\omega} = 0, \quad (24)$$

$$\frac{G_{-\omega}}{\omega} \frac{\partial}{\partial x} \Phi_{-\omega} = \frac{R_d}{R_b L} \left[ \frac{\Phi_{-\omega}}{\omega} I_1(-\omega, -\phi) + i e^{-i\Psi} I_2(-\omega, -\phi) \right].$$

To check the consistency of the four above equations, we consider the  $\omega$  dependence of several quantities. One can show that

$$f_{1\pm}(-\omega) = f_{1\pm}(\omega), \quad f_{2\pm}(-\omega) = f_{2\pm}(\omega), \\ g(-\omega) = -g(\omega) \quad (25)$$

As a result,

$$g_s(-\omega, -\phi) = -g_s(\omega, \phi), \quad f_s(-\omega, -\phi) = -f_s(\omega, \phi) \quad (26)$$

By comparing Eqs. (23) with Eq. (25), we can derive

$$G_{-\omega} = -G_\omega \quad (27)$$

Two cases can be considered:

(1)

$$\Phi_{-\omega} = \Phi_\omega, \quad I_1(-\omega, -\phi) = -I_1(\omega, \phi) \\ I_2(-\omega, -\phi) = I_2(\omega, \phi) \quad (28)$$

(2)

$$\Phi_{-\omega} = -\Phi_\omega, \quad I_1(-\omega, -\phi) = -I_1(\omega, \phi) \\ I_2(-\omega, -\phi) = -I_2(\omega, \phi) \quad (29)$$

For the case (1), the relations  $B(-\omega) = B(\omega)$  and  $C(-\omega) = C(\omega)$  hold, while for case (2) the relations  $B(-\omega) = -B(\omega)$  and  $C(-\omega) = -C(\omega)$  hold. For the case (1), since  $\Lambda_m(-\omega, -\phi) \neq \Lambda_m(\omega, \phi)$ , then it is impossible to satisfy  $I_1(-\omega, -\phi) = -I_1(\omega, \phi)$  and  $I_2(-\omega, -\phi) = I_2(\omega, \phi)$  simultaneously, thus, this case can not be realized. For the case (2),

$$\Lambda_m(-\omega, -\phi) = \Lambda_m(\omega, \phi), \quad (30)$$

is satisfied and this relation is consistent with  $I_1(-\omega, -\phi) = -I_1(\omega, \phi)$   $I_2(-\omega, -\phi) = -I_2(\omega, \phi)$ . Since  $\Phi(\omega) = -\Phi(\omega)$  is satisfied, we can show

$$\begin{aligned}\sin\theta(-\omega)\cos\psi(-\omega) &= -\sin\theta(\omega)\cos\psi(\omega), \\ \sin\theta(-\omega)\sin\psi(-\omega) &= -\sin\theta(\omega)\sin\psi(\omega).\end{aligned}\quad (31)$$

Then  $F_{-\omega} = -F_{\omega}$  is satisfied. This indicates the realization of the odd-frequency pairing state in DN. In the presence of  $h$ , *i.e.*, CP/DF/CP junctions, the admixture of even-frequency spin-singlet even-parity state is also present.

The Josephson current is given by

$$\frac{eIR}{\pi T_C} = i \frac{RTL}{4R_d T_C} \sum_{\uparrow, \downarrow, \omega} \frac{G_{\omega}^2}{\omega^2} \left( \Phi_{\omega} \frac{\partial}{\partial x} \Phi_{-\omega}^* - \Phi_{-\omega}^* \frac{\partial}{\partial x} \Phi_{\omega} \right), \quad (32)$$

where  $T$  is temperature, and  $R = R_b + R'_b + R_d$  is the total resistance of the junction. In the following, we fix  $R'_b = R_b$ ,  $T'_m = T_m$ , and choose  $\Delta(\phi) = \Delta e^{i\phi}$ . Here, we define  $\Delta_0$  as  $\Delta_0 \equiv \Delta(0)$ .

### III. RESULTS

#### A. CP / DN / CP junctions

First, we consider the temperature dependence of a maximum Josephson current  $I_C$  for  $Z = 10$  as shown in Fig.2. The magnitude of  $I_C$  is enhanced for large  $E_{Th}/\Delta_0$  and large  $R_d/R_b$ . It is enhanced at low temperatures in both cases (a) and (b). These features are consistent with conventional case of S/ DN /S junctions.

Next, we consider the dependence of  $I_C$  on  $Z$ , the transparency parameter at the interface. Figure 3 shows the temperature dependence of the critical Josephson current for  $R_d/R_b = 1$ . The magnitude of  $RI_C$  is enhanced for large  $Z$ , *i.e.*, low transparent interface for both  $E_{Th}/\Delta_0 = 0.1$  and 1. This result is specific for junctions between triplet superconductors, where proximity effect is enhanced by MARS formed at the interface. It is known that the degree of the influence of MARS on the charge transport becomes prominent for low transparent junctions with large  $Z^{15}$ . On the contrary, in S/DN/S junctions the maximum Josephson current is suppressed for large  $Z^{19}$ .

Next, we study the current-phase relation in order to examine the unusual proximity effect specific to CP/DN/CP junctions. Figure 4 shows the current-phase relation for  $Z = 10$  and  $R_d/R_b = 1$ . We find that the peak is shifted to  $\Psi > 0.5\pi$  at low temperatures, and this effect becomes rather strong in particular for large  $E_{Th}/\Delta_0$ . The result indicates that the magnitude of the Josephson current is enhanced by the proximity effect, and the Josephson current is not proportional to  $\sin\Psi$ . However this effect is not as pronounced as in the case of  $p_x$ -wave /DN/ $p_x$ -wave (P/DN/P) junctions<sup>19</sup>.

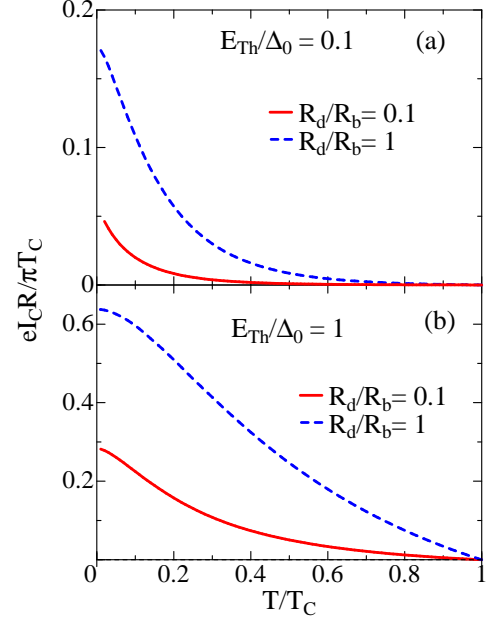


FIG. 2: (Color online) Temperature dependence of the maximum Josephson current for  $Z = 10$ . The solid lines are the results for  $R_d/R_b = 0.1$  and broken lines are the results for  $R_d/R_b = 1$ . (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

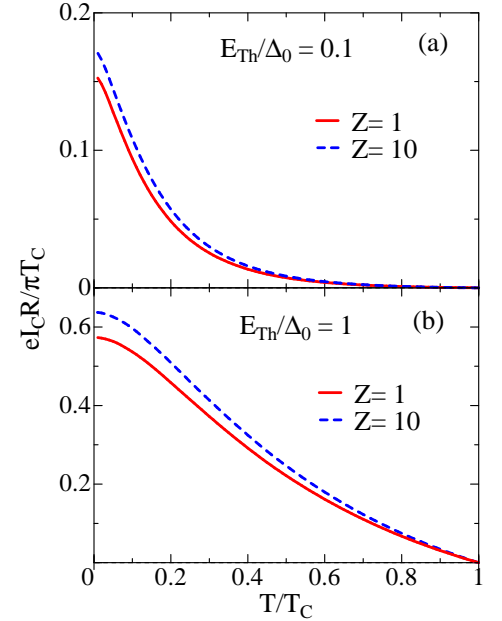


FIG. 3: (Color online) Temperature dependence of the maximum Josephson current for  $R_d/R_b = 1$ . The solid lines are the results for  $Z = 1$  and the broken lines are the results for  $Z = 10$ . (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

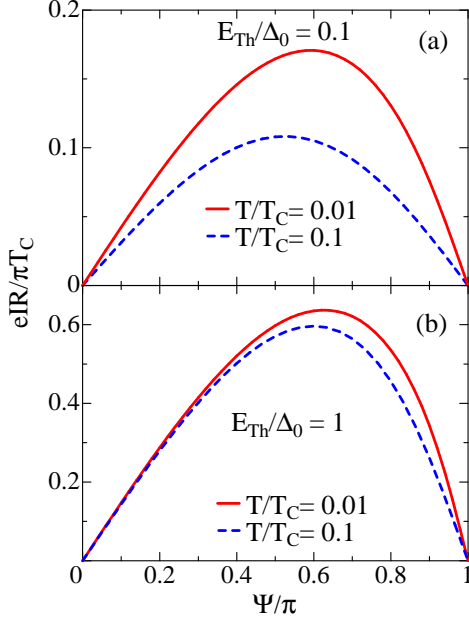


FIG. 4: (Color online) The current-phase relation for  $Z = 10$  and  $R_d/R_b = 1$ . The solid lines are the results for  $T/T_C = 0.01$  and the broken lines are the results for  $T/T_C = 0.1$ . (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

The dependence of the  $I_C$  on the length of DN is shown in Figure 5 for  $Z = 10$  and  $R_d/R_b = 1$ . We find that the  $I_C$  is proportional to  $\exp(-L/\xi)$  in agreement with existing theoretical results.

In order to compare our results with the existing theories, we also calculate  $I_C$  in S/DN/S junction and P/DN/P junctions<sup>27,28</sup>. The results are shown in Figure 6 for  $Z = 10$  and  $R_d/R_b = 1$ . We find that the magnitude of  $I_C$  in CP/DN/CP junction is larger than that in S/DN/S junction, and less than that in P/DN/P junction at low temperatures for both (a) and (b). These results indicate that the maximum Josephson current is enhanced due to the unusual proximity effect coexisting with MARS in CP/DN/CP junctions. However, it is known that MARS is induced only for the particle with injection angle  $\phi = 0$  in CP/DN/CP junctions, thus the  $I_C$  is smaller than in P/DN/P junctions. We also find that qualitative temperature dependence of the critical current in CP/DN/CP junctions is quite similar to that in S/DN/S junctions. The result is consistent with the experiment in  $Sr_2RuO_4$ - $Sr_3Ru_2O_7$  eutectic junctions<sup>38</sup>. As follows from these calculations, if we focus on the temperature dependence and current phase relation of the Josephson current of CP/DN/CP junctions, the obtained results are rather conventional.

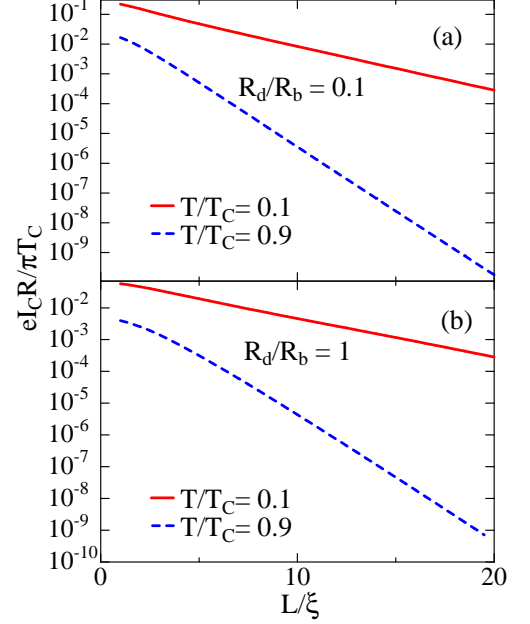


FIG. 5: (Color online) The critical current as a function of DN length for  $Z = 10$ . The solid lines are the results for  $T/T_C = 0.1$  and the broken lines are the results for  $T/T_C = 0.9$ . (a)  $R_d/R_b = 0.1$  and (b)  $R_d/R_b = 1$ .

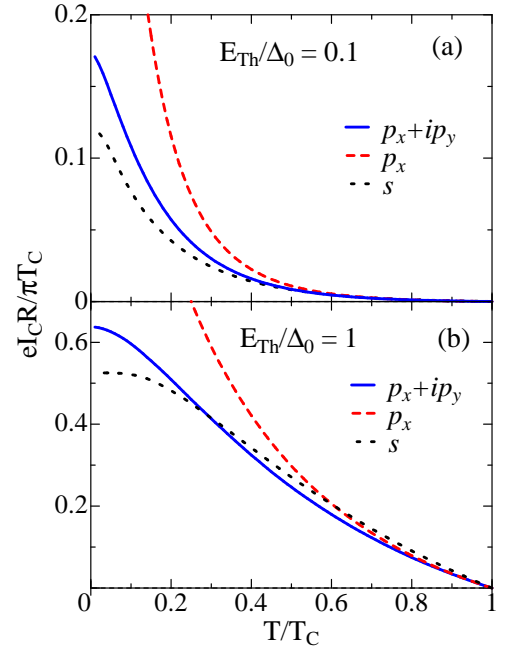


FIG. 6: (Color online) Temperature dependence of the critical current for  $Z = 10$  and  $R_d/R_b = 1$ . The solid lines are the results for CP/DN/CP junctions, the broken lines are the results for P/DN/P junctions, and the dot-lines are the results for S/DN/S junctions. (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

## B. CP / DF / CP junctions

It is well known that in S/DF/S junctions,  $0-\pi$  transition<sup>32,33</sup> can be induced. This phenomenon is due to the proximity effect specific to DF. In DF, the Cooper pairs have finite center of mass momentum and the pair amplitude is spatially oscillating. As a result, various interesting phenomena are predicted in these junctions<sup>29,34,35,36,37</sup>. The  $0-\pi$  transition is a typical example. It also exists in  $d(p)$ -wave superconductor / DF /  $d(p)$ -wave superconductor junctions<sup>28</sup>. Here we study the Josephson effect in CP / DF / CP junctions.

Figure 7 shows the temperature dependence of the critical current for  $Z = 10$  and  $R_d/R_b = 1$ . In all cases, the exchange field suppresses the magnitude of  $I_C$ . For  $E_{Th}/\Delta_0 = 0.1$  and  $h/\Delta_0 = 0.5$ , the non-monotonic temperature dependence of  $I_C$  is realized.

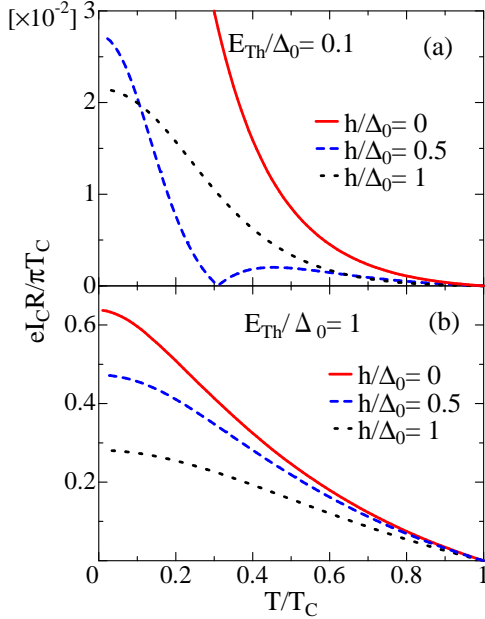


FIG. 7: (Color online) Temperature dependence of the critical current for  $Z = 10$  and  $R_d/R_b = 1$ . The solid lines are the results for  $h/\Delta_0 = 0$ , the broken lines are the results for  $h/\Delta_0 = 0.5$ , and the dot-lines are the results for  $h/\Delta_0 = 1$ . (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

To clarify that this non-monotonic temperature dependence originates from the  $0-\pi$  transition, we focus on the current phase relation as shown in Figure 8 for  $Z = 10$  and  $R_d/R_b = 1$  at  $T/T_C = 0.1$ . With the increase of the exchange field, the maximum of the Josephson current is shifted to  $\Psi < 0.5\pi$  for  $E_{Th}/\Delta_0 = 1$ . Especially, for  $E_{Th}/\Delta_0 = 0.1$ , the Josephson current changes its sign for  $h/\Delta_0 = 0.5$ . These results indicate that the exchange field induces the  $0-\pi$  transition in this case.

In Figure 9,  $I_C$  is plotted as a function of the length  $L$  of DF. In the presence of the exchange field  $h$  in DF, the

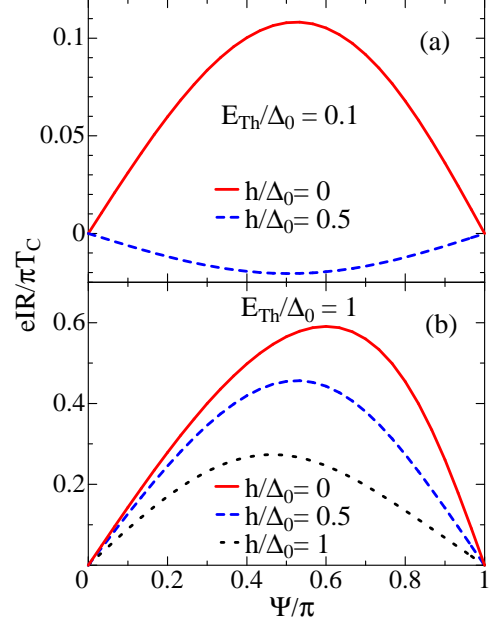


FIG. 8: (Color online) Current-phase relation for  $Z = 10$  and  $R_d/R_b = 1$  at  $T/T_C = 0.1$ . The solid lines are the results for  $h/\Delta_0 = 0$ , the broken lines are the results for  $h/\Delta_0 = 0.5$ , and the dot-line is the result for  $h/\Delta_0 = 1$ . (a)  $E_{Th}/\Delta_0 = 0.1$  and (b)  $E_{Th}/\Delta_0 = 1$ .

$I_C$  oscillates as a function of length of DF. The period of this oscillation becomes shorter with the increase of the magnitude of  $h$ .

We have shown that  $0-\pi$  transition also exists in the CP/DN/CP junctions. The nonmonotonic temperature dependence of  $I_C$  and the oscillatory dependence of  $I_C$  as a function of  $L$  are consistent with S/DN/S junctions or  $d(p)$ -wave superconductor / DF /  $d(p)$ -wave superconductor junctions<sup>28</sup>. It is shown that the  $0-\pi$  transition specific to DF junctions is robust against the change of the symmetry of the Cooper pair.

## IV. CONCLUSIONS

We have derived the generalized boundary conditions for DN (DF) / CP interface including the macroscopic phase of the superconductor. The Josephson effect in CP / DN (DF) / CP junctions has been studied by solving the Usadel equations with the above boundary conditions. Here, we choose the  $p_x + ip_y$ -wave as the symmetry of CP superconductor. The results obtained in the present paper can be summarized as follows:

1. It is shown that the symmetry of the induced pair wave function in DN due to the proximity effect is odd-frequency spin triplet  $s$ -wave. Josephson current is carried by the odd-frequency pairing state.



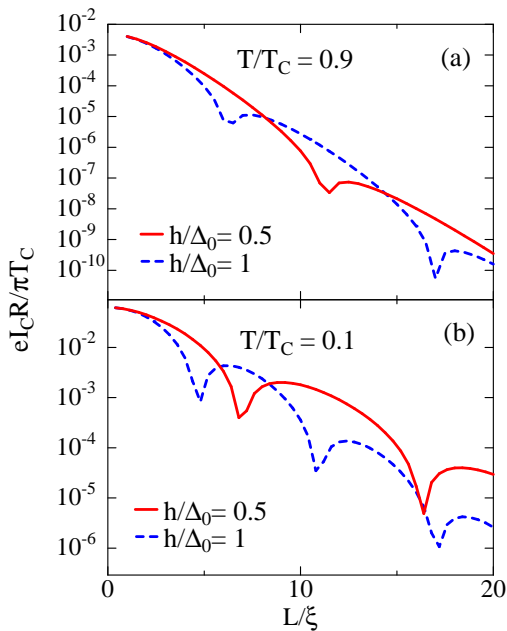


FIG. 9: (Color online) The critical current as a function of DF length for  $Z = 10$  and  $R_d/R_b = 1$ . (a)  $T/T_C = 0.9$  and (b)  $T/T_C = 0.1$ . The solid lines are the results for  $h/\Delta_0 = 0.5$  and the broken lines are the results for  $h/\Delta_0 = 1$ .

2. Almost all of the obtained results are qualitatively similar to those in S/DN/S junctions.  $I_C$  is proportional to  $\exp(-L/\xi)$  where  $L$  and  $\xi$  is the length of DN and coherence length of Cooper pair in DN, respectively. The temperature dependence of the maximum Josephson current in CP/DN/CP junction is qualitatively similar to that in the S/DN/S junctions.
3. Although the magnitude of the  $I_C$  is enhanced at low temperatures as compared to the corresponding S/DN/S junctions, this enhancement is not as strong as in the case of P/DN/P junctions.
4. In CP/DF/CP junctions, current phase relation changes drastically with the decrease of the temperature due to the  $0-\pi$  transition. The resulting  $I_C$  oscillates as a function of the width of DF. These properties are similar to those of S/DN/S junctions.

Recently, the Josephson effect in  $\text{Sr}_2\text{RuO}_4$ - $\text{Sr}_3\text{Ru}_2\text{O}_7$  eutectic junction is experimentally observed<sup>38</sup>. There is no qualitative difference of the temperature dependence as compared to that of S/DN/S junctions. The present theoretical result is consistent with this experiment. Surprisingly, although the proximity effect is unusual due to the presence of odd-frequency pairing state, the resulting Josephson current is not much different compared to the conventional junctions. The reason is that in the present case, the magnitude of the odd-frequency pair amplitude is small compared to that in P/DN/P junctions. Especially, the magnitude of the pair amplitude in DN for low Matsubara frequency in the present CP/DN/CP junctions is much smaller than that of P/DN/P junctions. It should be stressed that even though there is no qualitative difference between the actual experimentally observed Josephson current<sup>38</sup> and that in the S/DN/S junctions, it means neither absence of the spin-triplet pairing state in  $\text{Sr}_2\text{RuO}_4$  nor absence of the odd-frequency pairing state in DN.

In the present paper, we only focus on the diffusive limit. Recently, theory of proximity effect in the clean limit case is presented<sup>39</sup>. In such a case, the quasiclassical Green's function should be described by Eilenberger equation. It is an interesting issue to study the transition from clean limit to diffusive limits systematically.

## ACKNOWLEDGEMENT

T. Y. acknowledges support by JSPS Research Fellowships for Young Scientists. This work was supported by NAREGI Nanoscience Project, the Ministry of Education, Culture, Sports, Science and Technology, Japan, the Core Research for Evolutional Science and Technology (CREST) of the Japan Science and Technology Corporation (JST) and a Grant-in-Aid for the 21st Century COE "Frontiers of Computational Science". The computational aspect of this work has been performed at the Research Center for Computational Science, Okazaki National Research Institutes and the facilities of the Supercomputer Center, Institute for Solid State Physics, University of Tokyo and the Computer Center. This work is supported by Grant-in-Aid for Scientific Research on Priority Area "Novel Quantum Phenomena Specific to Anisotropic Superconductivity" (Grant No. 17071007) and B (Grant No. 17340106) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

<sup>1</sup> Y. Maeno, H. Hashimoto, K. Yoshida, S. Nishizaki, T. Fujita, J. G. Bednorz, and F. Lichtenberg : Nature **372** 532 (1994).  
<sup>2</sup> K. Ishida, H. Mukuda, Y. Kitaoka, K. Asayama, Z. Q. Mao, Y. Mori and Y. Maeno: Nature **396** (1998) 658.  
<sup>3</sup> G. M. Luke, Y. Fudamoto, K. M. Kojima, M. I. Larkin, J. Merrin, B. Nachumi, Y. J. Uemura, Y. Maeno, Z. Q. Mao,

Y. Mori, H. Nakamura and M. Sigrist: Nature **394** (1998) 558.

<sup>4</sup> A. P. Mackenzie and Y. Maeno, Rev. Mod. Phys. **75** 657 (2003).

<sup>5</sup> L. J. Buchholtz and G. Zwicknagl, Phys. Rev. B **23**, 5788 (1981).

<sup>6</sup> J. Hara and K. Nagai, Prog. Theor. Phys. **74**, 1237 (1986).



- <sup>7</sup> C. R. Hu, Phys. Rev. Lett. **72**, 1526 (1994); C. Bruder, Phys. Rev. B **41**, 4017 (1990).
- <sup>8</sup> Y. Tanaka and S. Kashiwaya, Phys. Rev. Lett. **74**, 3451 (1995); Y. Asano, Y. Tanaka and S. Kashiwaya, Phys. Rev. B **69**, 134501 (2004); S. Kashiwaya and Y. Tanaka, Rep. Prog. Phys. **63** 1641 (2000), Y. Tanaka, S. Kashiwaya : Phys. Rev. B **56** 892 (1997).
- <sup>9</sup> F. Laube, G. Goll, H.v. Löhneysen, M. Fogelström, and F. Lichtenberg, Phys. Rev. Lett. **84**, 1595 (2000); Z.Q. Mao, K.D. Nelson, R. Jin, Y. Liu, and Y. Maeno, Phys. Rev. Lett. **87**, 037003 (2001); M. Kawamura, H. Yaguchi, N. Kikugawa, Y. Maeno, H. Takayanagi, J. Phys. Soc. Jpn. **74**, 531 (2005).
- <sup>10</sup> M. Yamashiro, Y. Tanaka, and S. Kashiwaya, Phys. Rev. B **56**, 7847 (1997); M. Yamashiro, Y. Tanaka Y. Tanuma, and S. Kashiwaya, J. Phys. Soc. Jpn. **67**, 3224 (1998); M. Yamashiro, Y. Tanaka Y. Tanuma, and S. Kashiwaya, J. Phys. Soc. Jpn. **68**, 2019 (1999); C. Honerkamp and M. Sigrist, J. Low Temp. Phys. **111**, 895 (1998).
- <sup>11</sup> C. Honerkamp and M. Sigrist, Prog. Theor. Phys. **100**, 53 (1998); M. Yamashiro, Y. Tanaka, and S. Kashiwaya, J. Phys. Soc. Jpn. **67**, 3364 (1998); Y. Asano, Y. Tanaka, M. Sigrist, and S. Kashiwaya, Phys. Rev. B **67**, 184505 (2003).
- <sup>12</sup> R. Jin, Y. Liu, Z. Q. Mao and Y. Maeno: Europhys. Lett. **51** (2000) 341; A. Sumiyama, T. Endo, Y. Oda, Y. Yoshida, A. Mukai, A. Ono and Y. Onuki: Physica C **367** (2002) 129.
- <sup>13</sup> K. D. Nelson, Z. Q. Mao, Y. Maeno, and Y. Liu, Science **306** 1151 (2004).
- <sup>14</sup> Y. Asano, Y. Tanaka, M. Sigrist, and S. Kashiwaya Phys. Rev. B **71**, 214501 (2005)
- <sup>15</sup> Y. Tanaka and S. Kashiwaya, Phys. Rev. B **70**, 012507 (2004); Y. Tanaka, S. Kashiwaya and T. Yokoyama, Phys. Rev. B **71**, 094513 (2005); Y. Tanaka, Y. Asano, A. A. Golubov and S. Kashiwaya, Phys. Rev. B **72**, 140503(R) (2005).
- <sup>16</sup> W. Belzig, F. K. Wilhelm, C. Bruder, G. Schön, A.D. Zaikin, Superlattices and Microstructures, **25**, 1251 (1999); A. A. Golubov, M. Yu. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).
- <sup>17</sup> Y. Tanaka and A. Golubov and S. Kashiwaya, cond-mat/0610017; Y. Tanaka and A. Golubov, cond-mat/0609566.
- <sup>18</sup> Y. Asano, J. Phys. Soc. Jpn. **71**, 905 (2002).
- <sup>19</sup> Y. Asano, Y. Tanaka, and S. Kashiwaya Phys. Rev. Lett. **96**, 097007 (2006); Y. Asano, Y. Tanaka, T. Yokoyama and S. Kashiwaya, Phys. Rev. B **74** 064507 (2006).
- <sup>20</sup> Y. Asano, Phys. Rev. B **64**, 014511 (2001).
- <sup>21</sup> Y. Asano and K. Katabuchi, J. Phys. Soc. Jpn. **71**, 1974 (2002).
- <sup>22</sup> K.D. Usadel, Phys. Rev. Lett. **25** 507 (1970).
- <sup>23</sup> Yu. V. Nazarov, Phys. Rev. Lett. **73**, 1420(1994); Superlatt. Microstruct. **25** 1221(1999).
- <sup>24</sup> Y. Tanaka, Y.V. Nazarov and S. Kashiwaya, Phys. Rev. Lett. **90**, 167003 (2003). Y. Tanaka, Y. V. Nazarov, A. A. Golubov and S. Kashiwaya, Phys. Rev. B **69**, 144519 (2004).
- <sup>25</sup> Y. Tanaka and S. Kashiwaya, Phys. Rev. B **70**, 012507 (2004).
- <sup>26</sup> Y. Tanaka, S. Kashiwaya and T. Yokoyama, Phys. Rev. B **71**, 094513 (2005).
- <sup>27</sup> T. Yokoyama, Y. Tanaka, A. A. Golubov, and Y. Asano Phys. Rev. B **73**, 140504(R) (2006).
- <sup>28</sup> T. Yokoyama, Y. Tanaka, and A. A. Golubov, unpublished.
- <sup>29</sup> T. Yokoyama, Y. Tanaka, and A. A. Golubov, Phys. Rev. B **72**, 052512 (2005); Phys. Rev. B **73**, 094501 (2006).
- <sup>30</sup> K.K. Likharev, Rev. Mod. Phys. **51**, 101 (1979).
- <sup>31</sup> A. A. Golubov, M. Yu. Kupriyanov, and E. Il'ichev Rev. Mod. Phys. **76**, 411 (2004).
- <sup>32</sup> A. I. Buzdin, L. N. Bulaevskii, and S. Panjukov, JETP Lett. **35**, 178 (1982).
- <sup>33</sup> V. V. Ryazanov, V. A. Oboznov, A. Y. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Phys. Rev. Lett. **86**, 2427 (2001).
- <sup>34</sup> A. I. Buzdin, Rev. Mod. Phys. **77**, 935 (2005).
- <sup>35</sup> F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Rev. Mod. Phys. **77**, 1321 (2005).
- <sup>36</sup> T. Yokoyama, Y. Tanaka, A. A. Golubov, J. Inoue, and Y. Asano, Phys. Rev. B **71**, 094506 (2005).
- <sup>37</sup> T. Yokoyama and Y. Tanaka, C. R. Physique **7**, 136 (2006).
- <sup>38</sup> J. Hooper, M. Zhou, Z. Q. Mao, Y. Liu, R. Perry, and Y. Maeno, Phys. Rev. B **73**, 132510 (2006).
- <sup>39</sup> Y. Tanuma, Y. Tanaka and S. Kashiwaya, Phys. Rev. B **74**, 024506 (2006).